Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Note on Seminvariants of Systems of Partial Differential Equations.

By A. L. Nelson.

---

1. *Introduction.*

In the discussion of the projective differential geometry of a geometric configuration by means of Wilczynski's method, one of the necessary steps is the construction of a fundamental set of seminvariants, that is, such a set that any seminvariant whatever may be expressed in terms of them and their derivatives. In the cases where the completely integrable system of partial differential equations employed has one dependent variable, this construction has been accomplished by the reduction of the system of equations to its canonical form, the independent coefficients of which form are the fundamental set required. Let us illustrate the process by the case of plane nets.* The completely integrable system of equations for this case is the following:

$$y_{uu}=ay_u+by_v+cy,\quad y_{uv}=a'y_u+b'y_v+c'y,\quad y_{vv}=a''y_u+b''y_v+c''y. \tag{1}$$

The transformation

$$y=\lambda\overline{y} \tag{2}$$

yields a new system of equations of the same form as (1), with the coefficients

$$\left.\begin{aligned}
\overline{a}&=a-2\frac{\lambda_u}{\lambda}, & \overline{b}&=b, & \overline{c}&=c+a\frac{\lambda_u}{\lambda}+b\frac{\lambda_v}{\lambda}-\frac{\lambda_{uu}}{\lambda},\\
\overline{a}'&=a'-\frac{\lambda_v}{\lambda}, & \overline{b}'&=b'-\frac{\lambda_u}{\lambda}, & \overline{c}'&=c'+a'\frac{\lambda_u}{\lambda}+b'\frac{\lambda_v}{\lambda}-\frac{\lambda_{uv}}{\lambda},\\
\overline{a}''&=a'', & \overline{b}''&=b''-2\frac{\lambda_v}{\lambda}, & \overline{c}''&=c''+a''\frac{\lambda_u}{\lambda}+b''\frac{\lambda_v}{\lambda}-\frac{\lambda_{vv}}{\lambda}.
\end{aligned}\right\} \tag{3}$$

The integrability conditions enable us to find a function $p(u, v)$, such that $p_u=a+b'$, $p_v=a'+b''$. We find also that under the transformation (2) these combinations become

$$\overline{p_u}=p_u-3\frac{\lambda_u}{\lambda},\quad \overline{p}=p_v-3\frac{\lambda_v}{\lambda}.$$

---

* Cf. E. J. Wilczynski, "One-Parameter Families and Nets of Plane Curves," *Trans. Amer. Math. Soc.*, Vol. XII (1911), pp. 473-510.

If, therefore, λ be so chosen that

$$\frac{\lambda_u}{\lambda}=\tfrac{1}{3}p_u, \quad \frac{\lambda_v}{\lambda}=\tfrac{1}{3}p_v,$$

we get the special unique form of (1), which shall be indicated by capital letter coefficients, characterized by the relations $A+B'=0$, $A'+B''=0$. This is the canonical form of (1). We list its coefficients for later comparison.

$$\left.\begin{array}{lll}
A\ =\tfrac{1}{3}(\ a-2b'\ ), & B\ =b, & C\ =c\ +\tfrac{2}{9}a^2\ +\tfrac{1}{9}ab'+\tfrac{1}{3}a'b\ +\tfrac{1}{3}bb'' \\
 & & \qquad -\tfrac{1}{3}a_u\ -\tfrac{1}{3}b'_u-\tfrac{1}{9}b'^2, \\
A'\ =\tfrac{1}{3}(2a'-\ b''), & B'\ =\tfrac{1}{3}(2b'-\ a\ ), & C'\ =c'\ +\tfrac{2}{9}aa'+\tfrac{5}{9}a'b'+\tfrac{2}{9}b'b'' \\
 & & \qquad -\tfrac{1}{9}ab''-\tfrac{1}{3}a_v-\tfrac{1}{3}b'_v, \\
A''=a'', & B''=\tfrac{1}{3}(\ b''-2a'), & C''=c''+\tfrac{2}{9}b''^2+\tfrac{1}{9}a'b''+\tfrac{1}{3}a''b'+\tfrac{1}{3}aa'' \\
 & & \qquad -\tfrac{1}{3}b''_v\ -\tfrac{1}{3}a'_v-\tfrac{1}{9}a'^2.
\end{array}\right\}\quad(4)$$

Only seven of these coefficients are independent. They are the fundamental seminvariants. For, consider any system of equations of the form (1), to which we shall refer as (s), and any other system (t), obtained from it by any transformation of the type (2). Let (s) be reduced to its canonical form (S), and (t) to its canonical form (T). The coefficients of (S) are the same combinations of the coefficients of (s) as the corresponding coefficients of (T) are of those of (t). But, since the canonical form is unique, the corresponding coefficients of (S) and (T) are identical. Hence they are seminvariants. Moreover, any seminvariant whatever, I, is a function of the original coefficients of (s), *i. e.*, $I=I(a, b, \ldots, c'', a_u, a_v, \ldots)$. Since it is a seminvariant, it must be identical with the same function of the corresponding coefficients of any system of equations obtained from (s) by a transformation (2). In particular, $I=I\ (A, B, \ldots, C'', A_u, A_v, \ldots)$. Therefore I is a function of the seven independent coefficients of the canonical form and of their derivatives. Hence these coefficients are a fundamental set of seminvariants.

The determination of this canonical form depends upon the ability to find such a function p. Green* has shown that this can be done for completely integrable systems with one dependent variable and n independent variables, provided certain conditions are fulfilled. However, the expressions for p_u and p_v, which, with their derivatives, must occur in the coefficients of the canonical

---

*G. M. Green, "The Linear Dependence of Functions of Several Variables, and Completely Integrable Systems of Homogeneous Linear Partial Differential Equations," *Trans. Amer. Math. Soc.*, Vol. XVII (1916), pp. 483-516.

form, are not always simple, and as a result the seminvariants which come from this regulation method are sometimes rather complicated when expressed in terms of the original coefficients. We proceed to indicate how (in most cases, at least) an alternative set of simpler seminvariants may be computed.

2. *The Effect of Transformation* (2) *in the General Case.*

We assume that the completely integrable system of equations (corresponding to (1)), to which we shall refer as (1′), has one dependent variable, y,* and n independent variables, $u_1, \ldots, u_n$, $(n>1)$. It will then consist of a certain number, p, of equations which express certain p derivatives of y in terms of q chosen derivatives (which Green has called the primary derivatives). Let us also assume that among the q primary derivatives are none of order higher than the order of the lowest of the p derivatives in the left members of the p equations.

The transformation (2) gives rise to the following expressions of the various derivatives of y in terms of the derivatives of $\overline{y}$.

$$\left.\begin{aligned} y &= \lambda\overline{y}, \\ y_{u_i} &= \lambda\overline{y}_{u_i} + \lambda_{u_i}\overline{y}, \quad (i=1, 2, \ldots, n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ y^{\,l_1}_{u_1}\cdots{}^{l_n}_{u_n} &= \sum_{\substack{p_1=0,\ldots,l_1 \\ \cdots\cdots\cdots \\ p_n=0,\ldots,l_n}} \binom{l_1}{p_1}\cdots\binom{l_n}{p_n}\lambda^{\,p_1}_{u_1}\cdots{}^{p_n}_{u_n}\,\overline{y}^{\,l_1-p_1}_{u_1}\cdots{}^{l_n-p_n}_{u_n}, \end{aligned}\right\} \tag{2′}$$

where

$$\binom{l_j}{p_j} \equiv \frac{l_j!}{p_j!\,(l_j-p_j)!} \quad\text{and}\quad y^{\,l_1}_{u_1}\cdots{}^{l_n}_{u_n} \equiv \frac{\partial^{l_1+\cdots+l_n}y}{\partial u_1^{l_1}\cdots\partial u_n^{l_n}}.$$

If we substitute (2′) in the system (1′), and arrange the resulting equations properly, we get a new system of the same form as (1′), in which the dependent variable is $\overline{y}$. The coefficients of this new system, which, in collected form, we shall refer to as (3′), are equal to the corresponding coefficients of (1′) plus linear combinations of fractions whose numerators are derivatives of λ, and whose common denominator is λ. Each coefficient (3′) receives additional terms from those y-derivatives in its equation which come by differentiation from the y-derivative belonging to this coefficient. (Thus, for example, in (3), $\overline{b}'$, the coefficient of $\overline{y}_v$, can receive only one additional term, $-\frac{\lambda_u}{\lambda}$, which comes from y_{uv}; $\overline{c}'$ receives one λ-fraction from each of the derivatives y_u, y_v, y_{uv}.)

---

* This assumption is merely for the sake of simplifying the discussion. The process may be readily extended to cover the cases where the number of dependent variables is greater than one.

We arrange these coefficients (3′) in classes as follows: In the zero-th class shall be placed those coefficients in each row which are coefficients of those primary derivatives having the same order as the derivative in the left member of the equation belonging to that row. The i-th class shall consist of the coefficients of those primary derivatives in each row of order i less than the derivative in the left member of the equation belonging to that row. (Thus, in table (3), the coefficients in the first and second columns are of the first class; those in the third column are of the second class.) The zero-th class coefficients, if present, must be seminvariants, since their y-derivatives can not yield by differentiation any other y-derivative in their equations. The first class coefficients will be altered, if at all, by the addition of multiples of λ-fractions of the first order, the coefficients of which λ-fractions are integers or integral multiples of zero-th class coefficients (3′). In general, the i-th class coefficients will be altered by the addition of linear combinations of λ-fractions of order not higher than i. Each one of the coefficients of these λ-fractions will be an integer or an integral multiple of some one coefficient (3′) of classes 0 to i—1.

3. *Pseudo-Canonical Forms of* (1′).

Now it very frequently happens that *under suitable restrictions* we may make n first class coefficients (3′) vanish by a proper choice of λ. For example, in (3), under the assumption $a'_u=b'_v$, we might have made $\bar{a}'=\bar{b}'=0$. Or we might have made $\bar{a}=\bar{b}''=0$, provided $a_v=b''_u$. If we can do this, under the necessary restriction the new form of the completely integrable system determined by this choice of λ is unique, because of the nature of the equations in λ. We shall call this unique form a *pseudo-canonical form of* (1′). Moreover, the necessary restrictions are always seminvariantive ones. For, consider certain n such first class coefficients,

$$\bar{\alpha}_i=\alpha_i-k_i\frac{\partial}{\partial u_i}\log\lambda,\qquad (i=1,2,\ldots,n),\tag{5}$$

where the k_i are at worst integral multiples of seminvariant coefficients of (1′). In order to choose λ to satisfy the equations

$$\alpha_i-k_i\frac{\partial}{\partial u_i}\log\lambda=0,\qquad (i=1,2,\ldots,n),\tag{6}$$

the assumptions necessary are

$$\frac{\partial}{\partial u_i}\left(\frac{\alpha_j}{k_j}\right)=\frac{\partial}{\partial u_j}\left(\frac{\alpha_i}{k_i}\right),\qquad (i,j=1,2,\ldots,n),\tag{7}$$

which are easily seen to be seminvariantive from (5).

Let us assume the necessary restrictions (7) and choose λ in accordance with (6). However, there is a precaution to be observed in substituting for λ. Whenever a cross-derivative,

$$\frac{1}{\lambda}\cdot\frac{\partial^{p_1+\dots+p_n}\lambda}{\partial u_1^{p_1}\dots\partial u_n^{p_n}}\quad(\text{at least two } p\text{'s} \neq 0),$$

occurs, equations (7) afford us a choice in the substitution. We must, throughout a coefficient (3′), replace such cross-derivatives by the proper derivatives of the *same* members of equations (7). The reason for this precaution will become apparent in the next section. Then, since the restrictions are seminvariantive, and since the pseudo-canonical form is unique, the coefficients of this form are seminvariants, by precisely the same argument as was made in Section 1 for the coefficients of the canonical form in the case of plane nets.

4. *The Coefficients of a Pseudo-Canonical Form are a Fundamental Set of Seminvariants.*

Now, if the restrictions (7) be removed, the pseudo-canonical form will no longer exist, *but the combinations we have obtained are still seminvariants.* For, let us form the same combinations of the coefficients (3′). If we do this under the assumption (7), of course all traces of λ will disappear. That this must also be true without these assumptions is evident from the following considerations:

Let any one of these combinations be denoted by $f(a, b, \dots)$. It is composed of coefficients of (1′) of various classes. However, from the nature of the choice of λ, the only derivatives which appear in $f(a, b, \dots)$ are derivatives of coefficients of the first class. Now, let us form the same combination of the coefficients (3′), $f(\bar{a}, \bar{b}, \dots)$, and examine the resulting expression. It may acquire derivatives of coefficients of (1′) only from the following sources: 1. Those derivatives which appeared in $f(a, b, \dots)$ will remain. These, as we have noted, are derivatives of coefficients of (1′) of the first class only. 2. Those arguments, $\bar{a}, \bar{b}, \dots$, which are of the first class (whose expressions are given by (3′)), may have as coefficients of λ-fractions zero-th class coefficients of (1′). Hence, when the expressions (3′) for these arguments are substituted in $f(\bar{a}, \bar{b}, \dots)$, the resulting expression may contain derivatives of these zero-th class coefficients. It thus appears that the only derivatives of coefficients of (1′) of any class except the zero-th which may occur are those from the first source. Moreover, we have taken the precaution that among these derivatives in $f(a, b, \ldots\ldots)$ shall appear only one member of any of equations (7) and its derivatives. Hence, if we collect the expression resulting from substitution of the values (3′) for the arguments in $f(\bar{a}, \bar{b}, \ldots\ldots)$, the collection being with respect to the λ-fractions of various orders, no one of these fractions may have as coefficient an expression which vanishes by virtue of assumptions (7). Therefore, since in the resulting expression all terms in λ cancelled under assumptions (7), the same thing must happen without these assumptions.

It is to be observed that in a pseudo-canonical form, there are $pq-n$ non-vanishing coefficients, precisely the number of independent coefficients in Green's generalized canonical form. Moreover, these coefficients are independent, since no two arise from the same coefficient of the original system (1′). From this it may be argued that the seminvariant coefficients of a pseudo-canonical form are a fundamental set.

5.* *Seminvariants Arising from Cross-Derivatives of log λ.*

In certain cases there is a lack of definiteness in the formation of seminvariants by the process outlined above. As mentioned previously, whenever a λ-fraction involving a cross-derivative of λ occurs in a coefficient (3′), either member of the proper equation of (7) may be used in forming the seminvariant corresponding to this coefficient. Indeed, certain expressions of the type

$$\frac{m\dfrac{\partial}{\partial u_i}\left(\dfrac{\alpha_j}{k_j}\right)+n\dfrac{\partial}{\partial u_j}\left(\dfrac{\alpha_i}{k_i}\right)}{m+n}$$

may be used instead of $\dfrac{\partial}{\partial u_i}\left(\dfrac{\alpha_j}{k_j}\right)$ or $\dfrac{\partial}{\partial u_j}\left(\dfrac{\alpha_i}{k_i}\right)$.

A coefficient of the type in question may be put in the form

$a_0+\sum\limits_{\substack{i,j=1,\ldots\ldots,n\\ i<j}} a^{(i,j)}D^{(i,j)}+$sum of terms of type $aD^{(i,j)}{}_{u_1\ldots\ldots u_n}^{l_1\quad l_n}$, $(l_1,\ldots\ldots, l_n=0, 1, \ldots\ldots)$, where $D^{(i,j)}\equiv\dfrac{\partial^2}{\partial u_i\partial u_j}\log\lambda$. Let us denote one of the possible substitutions for $D^{(i,j)}$ by $[D^{(i,j)}]$, and the other by $\{D^{(i,j)}\}$. We wish to find what expressions of the form

$$F^{(i,j)}\equiv\frac{m^{(i,j)}[D^{(i,j)}]+n^{(i,j)}\{D^{(i,j)}\}}{m^{(i,j)}+n^{(i,j)}}, \qquad (m^{(i,j)}+n^{(i,j)}\neq 0), \tag{8}$$

---

* Section 5 was added at Professor Wilczynski's suggestion.

may be used to replace $D^{(i,j)}$. We shall find it convenient to use the notation

$$(M^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} D^{(i,j)}_{\substack{q_1 \ldots q_n \\ u_1 \ldots u_n}}) \equiv m^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} [D^{(i,j)}_{\substack{q_1 \ldots q_n \\ u_1 \ldots u_n}}] + n^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} \{D^{(i,j)}_{\substack{q_1 \ldots q_n \\ u_1 \ldots u_n}}\}, \tag{9}$$

and it is easily verified that

$$(M^{(i,j)}D^{(i,j)})_{u_k} = (M^{(i,j)}D^{(i,j)}_{u_k}) + (M^{(i,j)}_{u_k}D^{(i,j)}), \text{ etc.,} \qquad (i,\, j,\, k = 1, \ldots, n).$$

Now, if we denote by $\bar{\theta} = \theta + \delta\theta$ the same combination of the transformed coefficients (3′) as θ is of the original coefficients of (1′), we see from (5) that $\overline{[D^{(i,j)}]} = [D^{(i,j)}] - D^{(i,j)}$, $\overline{\{D^{(i,j)}\}} = \{D^{(i,j)}\} - D^{(i,j)}$. If, therefore, $F^{(i,j)}$ is to replace $D^{(i,j)}$, it must happen that

$$\delta F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}} + D^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}} = 0, \qquad (l_1, \ldots, l_n = 0, 1, \ldots).$$

In order to compute $\delta F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}}$, let us successively differentiate the equations

$$(m^{(i,j)} + n^{(i,j)})F^{(i,j)} = (M^{(i,j)}D^{(i,j)}),$$

which result from (8) and (9). We obtain

$$\sum^{\substack{p_1=0,\, \ldots,\, l_1 \\ \cdots\cdots\cdots \\ p_n=0,\, \ldots,\, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} F^{(i,j)}_{\substack{l_1-p_1 \ldots l_n-p_n \\ u_1 \ldots u_n}}$$

$$= \sum^{\substack{p_1=0,\, \ldots,\, l_1 \\ \cdots\cdots\cdots \\ p_n=0,\, \ldots,\, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (M^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} D^{(i,j)}_{\substack{l_1-p_1 \ldots l_n-p_n \\ u_1 \ldots u_n}}).$$

From these equations

$$F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}} = \frac{1}{m^{(i,j)} + n^{(i,j)}} \left\{ \sum^{\substack{p_1=0,\, \ldots,\, l_1 \\ \cdots\cdots\cdots \\ p_n=0,\, \ldots,\, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (M^{(i,j)}_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} D^{(i,j)}_{\substack{l_1-p_1 \ldots l_n-p_n \\ u_1 \ldots u_n}}) \right.$$

$$- \sum_{s=1}^{n} \sum^{\substack{p_r=1,\, \ldots,\, l_r \\ r \neq s}} \binom{l_1}{p_1} \cdots \binom{l_{s-1}}{p_{s-1}} \binom{l_{s+1}}{p_{s+1}} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \ldots p_{s-1} \; p_{s+1} \ldots p_n \\ u_1 \ldots u_{s-1} \; u_{s+1} \ldots u_n}} F^{(i,j)}_{\substack{l_1-p_1 \ldots l_s \ldots l_n-p_n \\ u_1 \ldots u_s \ldots u_n}}$$

$$\left. - \sum^{\substack{p_1=1,\, \ldots,\, l_1 \\ \cdots\cdots\cdots \\ p_n=1,\, \ldots,\, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \ldots p_n \\ u_1 \ldots u_n}} F^{(i,j)}_{\substack{l_1-p_1 \ldots l_n-p_n \\ u_1 \ldots u_n}} \right\},$$

and $\overline{F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}}}$ will have a similar expression with $\overline{m^{(i,j)}}$, $\overline{n^{(i,j)}}$, $\overline{M^{(i,j)}}$, $\overline{D^{(i,j)}}$ and $\overline{F^{(i,j)}}$, replacing $m^{(i,j)}$, $n^{(i,j)}$, $M^{(i,j)}$, $D^{(i,j)}$ and $F^{(i,j)}$, respectively. Use of the relation

$$\delta F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}} = \overline{F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}}} - F^{(i,j)}_{\substack{l_1 \ldots l_n \\ u_1 \ldots u_n}}$$

gives us, after considerable labor,

$$
\left.\begin{aligned}
&\delta F^{(i,j)}_{\substack{l_1 \quad\; l_n\\ u_1\dots u_n}} + D^{(i,j)}_{\substack{l_1 \quad\; l_n\\ u_1\dots u_n}} = \\
\text{(a)}\quad & \sum^{\substack{p_1=0,\,\dots,\,l_1\\ \dots\dots\dots\\ p_n=0,\,\dots,\,l_n}} \binom{l_1}{p_1}\cdots\binom{l_n}{p_n}\left(\delta\,\frac{\overset{(i,j)}{m}_{\substack{p_1 \quad p_n\\ u_1\dots u_n}}}{m^{(i,j)}+n^{(i,j)}}\,[D^{(i,j)}_{\substack{l_1-p_1\; l_n-p_n\\ u_1\dots u_n}}]\right.\\
&\qquad\qquad\qquad\qquad\left. +\,\delta\,\frac{\overset{(i,j)}{n}_{\substack{p_1 \quad p_n\\ u_1\dots u_n}}}{m^{(i,j)}+n^{(i,j)}}\,\{D^{(i,j)}_{\substack{l_1-p_1\; l_n-p_n\\ u_1\dots u_n}}\}\right)\\
\text{(b)}\quad & -\sum_{s=1}^{n}\ \sum^{\substack{p_r=1,\,\dots,\,l_r\\ r\neq s}} \binom{l_1}{p_1}\cdots\binom{l_{s-1}}{p_{s-1}}\binom{l_{s+1}}{p_{s+1}}\cdots\binom{l_n}{p_n}\,\delta\,\frac{(m^{(i,j)}+n^{(i,j)})_{\substack{p_1 \quad p_{s-1} \; p_{s+1} \quad p_n\\ u_1\dots u_{s-1}\; u_{s+1}\dots u_n}}}{m^{(i,j)}+n^{(i,j)}}\\
&\qquad\qquad\qquad\qquad\cdot F^{(i,j)}_{\substack{l_1-p_1\; l_s \quad l_n-p_n\\ u_1\dots u_s\dots u_n}}\\
\text{(c)}\quad & -\sum^{\substack{p_1=1,\,\dots,\,l_1\\ \dots\dots\dots\\ p_n=1,\,\dots,\,l_n}} \binom{l_1}{p_1}\cdots\binom{l_n}{p_n}\,\delta\,\frac{(m^{(i,j)}+n^{(i,j)})_{\substack{p_1 \quad p_n\\ u_1\dots u_n}}}{m^{(i,j)}+n^{(i,j)}}\cdot F^{(i,j)}_{\substack{l_1-p_1\; l_n-p_n\\ u_1\dots u_n}}\\
\text{(d)}\quad & -\sum_{s=1}^{n}\ \sum^{\substack{p_r=1,\,\dots,\,l_r\\ r\cdot s}} \binom{l_1}{p_1}\cdots\binom{l_{s-1}}{p_{s-1}}\binom{l_{s+1}}{p_{s+1}}\cdots\binom{l_n}{p_n}\,\frac{\overline{(m^{(i,j)}}+\overline{n^{(i,j)})}_{\substack{p_1 \quad p_{s-1} \; p_{s+1} \quad p_n\\ u_1\dots u_{s-1}\; u_{s+1}\dots u_n}}}{\overline{m^{(i,j)}}+\overline{n^{(i,j)}}}\\
&\qquad\qquad\qquad\qquad(\delta F^{(i,j)}_{\substack{l_1-p_1\; ls \quad l_n-p_n\\ u_1\dots u_s\dots u_n}}+D^{(i,j)}_{\substack{l_1-p_1\; l_s \quad l_n-p_n\\ u_1\dots u_s\dots u_n}})\\
\text{(e)}\quad & -\sum^{\substack{p_1=1,\,\dots,\,l_1\\ \dots\dots\dots\\ p_n=1,\,\dots,\,l_n}} \binom{l_1}{p_1}\cdots\binom{l_n}{p_n}\,\frac{\overline{(m^{(i,j)}}+\overline{n^{(i,j)})}_{\substack{p_1 \quad p_n\\ u_1\dots u_n}}}{\overline{m^{(i,j)}}+\overline{n^{(i,j)}}}\,(\delta F^{(i,j)}_{\substack{l_1-p_1\; l_n-p_n\\ u_1\dots\dots u_n}}+D^{(i,j)}_{\substack{l_1-p_1\; l_n-p_n\\ u_1\dots\dots u_n}}).
\end{aligned}\right\}\text{(10)}
$$

It is evident that (a) will vanish if

$$
\delta\,\frac{\overset{(i,j)}{m}_{\substack{p_1 \quad p_n\\ u_1\dots u_n}}}{m^{(i,j)}+n^{(i,j)}}=\delta\,\frac{\overset{(i,j)}{n}_{\substack{p_1 \quad p_n\\ u_1\dots u_n}}}{m^{(i,j)}+n^{(i,j)}}=0\quad(p_1=0,\dots,l_1;\dots;\ p_n=0,\dots,l_n).\quad\text{(11)}
$$

The conditions (11) imply

$$
\delta\,\frac{(m^{(i,j)}+n^{(i,j)})_{\substack{p_1 \quad p_n\\ u_1\dots u_n}}}{m^{(i,j)}+n^{(i,j)}}=0\quad(p_1=0,\dots,l_1;\dots;\ p_n=0,\dots,l_n),
$$

so that (b) and (c) vanish as a result. Moreover, (11) also imply

$$
\delta F^{(i,j)}_{\substack{l_1-p_1\; l_s \quad l_n-p_n\\ u_1\dots u_s\dots u_n}}+D^{(i,j)}_{\substack{l_1-p_1\; l_s \quad l_n-p_n\\ u_1\dots u_s\dots u_n}}=0,
$$

($p_1=0,\dots,l_1;\dots;\ p_n=0,\dots,l_n$; except that not all p's$=0$; $s=1,\dots,n$), by virtue of (10) for successive values of the l's. Hence (d) and (e) also vanish as a result of (11). Therefore, *a sufficient condition that $F^{(i,j)}$ may* *replace* $[D^{(i,j)}]$ *or* $\{D^{(i,j)}\}$ *in any seminvariant without affecting its seminvariance, is that for each pair* $(i, j=1, \ldots, n;\ i<j)$,

$$\frac{\overset{(i,j)}{m}{}^{p_1 \quad p_n}_{u_1 \ldots u_n}}{m^{(i,j)}+n^{(i,j)}} \text{ and } \frac{\overset{(i,j)}{n}{}^{p_1 \quad p_n}_{u_1 \ldots u_n}}{m^{(i,j)}+n^{(i,j)}} (m^{(i,j)}+n^{(i,j)} \neq 0;\ p_1=0, \ldots, l_1;\ \ldots;\ p_n=0, \ldots, l_n)$$

be seminvariants, where l_r *is the highest order of a derivative of* $[D^{(i,j)}]$ *(or* $\{D^{(i,j)}\}$*) with respect to* u_r $(r=1, \ldots, n)$ *occurring in the original seminvariant.*

6. *Application of the New Method to Special Cases.*

It should be noted that while we have indicated a large class of expressions which may be used in the place of $[D^{(i,j)}]$ or $\{D^{(i,j)}\}$, if we wish the simplest set of seminvariants we will set $m^{(i,j)}=1$, $n^{(i,j)}=0$, or $m^{(i,j)}=0$, $n^{(i,j)}=1$.

Let us now illustrate this method of computing seminvariants by the case of plane nets. We assume temporarily $a'_u=b'_v$, and take $\lambda_u/\lambda=b'$, $\lambda_v/\lambda=a'$. Substitution of these values in (3) yields the fundamental set of seminvariants

$$\left.\begin{array}{lll} A_1=a-2b', & B_1=b, & C_1=c\ +ab'+a'b-b'_u-b'^2,^* \\ A'_1=0, & B'_1=0, & C'_1=c'\ +a'b'-b'_v, \text{ or } c'+a'b'-a'_u, \\ A''_1=a'', & B''_1=b''-2a', & C''_1=c''+a''b'+a'b''-a'_v-a'^2,^* \end{array}\right\} \quad (12)$$

which are considerably simpler than those of (4). A further advantage of the new process over the old becomes evident when we discuss the invariants of (1). Of the seminvariants (4), only two, A'' and B, are invariants. Of the seminvariants (12), *five* are invariants, viz., those just mentioned and in addition $C_1=\mathfrak{C}$, $C'_1=K$ (or H), $C''_1=\mathfrak{C}''$. That this advantage can be guaranteed in general, however, is not evident. The simplicity of the new process might perhaps give it a better chance than the old to yield a larger number of seminvariants which are invariants as well. Another pseudo-canonical form, viz., that determined by $\lambda_u/\lambda=\frac{1}{2}a$, $\lambda_v/\lambda=\frac{1}{2}b''$, yields a fundamental set of seminvariants which are simpler than those of (4), but not quite so simple as those of (12). Only two of these are invariants.

In the case of conjugate systems of curves on a curved surface,† the new process (for a certain choice of λ) gives seminvariants which are very much simpler than those given by the regulation method. Moreover, three of the five are seminvariants, as against one out of five for the old method. In the case of developable surfaces,* besides yielding simpler seminvariants, the new method gives two invariants, while the other gives none.

---

* It happens that C_1 and C_1'' may be still further simplified by use of the integrability conditions.

† G. M. Green, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVII (1915), pp. 215-246.

It seems evident that the method we have outlined will be of even greater value in the cases of higher dimensional configurations. For example, the simplification due to the new process for the case of curvilinear coordinates in n dimensions † is seen by noticing that it would replace, for every i,

$$f^{(i)} = \sum_{j=1}^{n} a_j^{(j,\,i)}$$

by some single coefficient $a_j^{(j,\,i)}$.

ANN ARBOR, MICH., *January* 2, 1918.

---

* W. W. Denton, *Trans. Amer. Math, Soc.*, Vol. XIV (1913), pp. 175–208.

† G. M. Green, "The Linear Dependence of Functions," etc. *loc. cit.* Cf. especially Section 7, and equations (21).